Development of an Energy Conserving Numerical Scheme Suitable for Geodesic Grids

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An abbreviated history......

Masuda and Ohnishi (1986) integrate the vorticity-divergence form of the shallow-water equations on a geodesic grid.

Randall (1994) shows that the unstaggered vorticity-divergence discretization does a better job than the conventional A- through E-grid staggerings in terms of geostrophic adjustment. This is the Z-grid.

Heikes and Randall (1995a,b) implement a multi-grid solver to mitigate the expense of inverting the elliptic equations required in the vorticity-divergence formulation.

Ringler et al. (2000) incorporate this formulation into a full physics AGCM.

Our Goal

Develop a numerical scheme that conserves certain quantities, such as total energy and potential enstrophy, while maintaining the superior simulation of geostrophic adjustment.

Grid Staggering

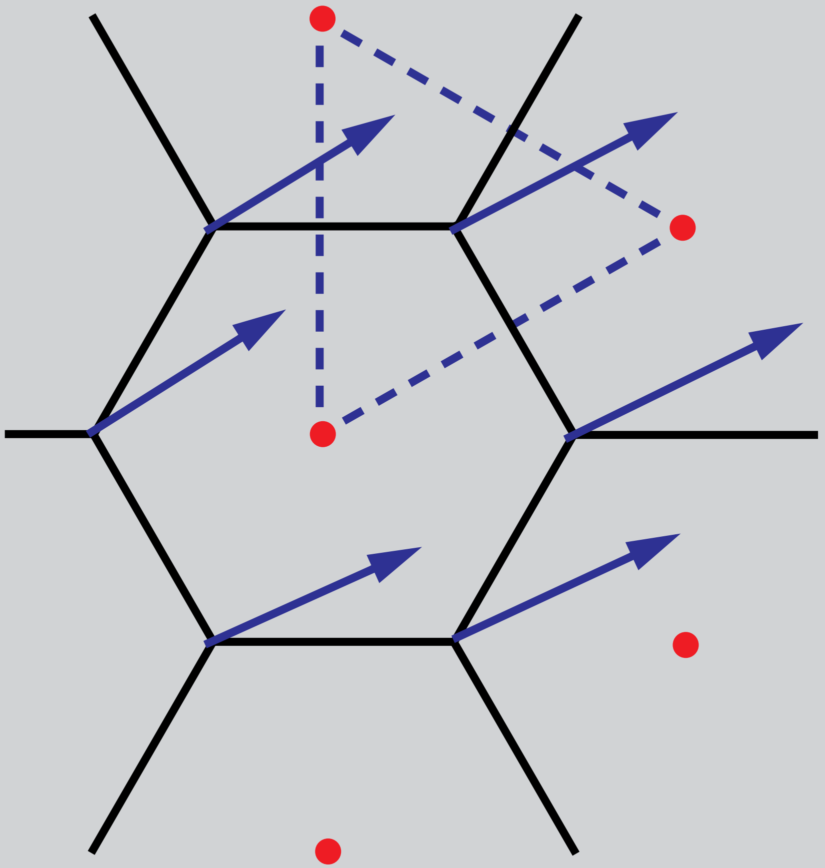
All scalars at grid cell centers
- **Mass**
- **Kinetic Energy**
- **Vorticity**
- **Divergence**

All vectors at grid cell corners
- **Velocity**
- **Gradients of scalar fields**

area of each momentum point

is $\frac{1}{2}$

the area of each mass point



Methodology

Write down the discrete shallow-water equations by analogy to the continuous equations.

Choose a form for one of the discrete operators. We will choose the form of the divergence operator.

By enforcing conservation of kinetic energy under the process of advection, determine the form of the discrete gradient operator.

By enforcing conservation of total energy, determine the discrete form of kinetic energy.

The Shallow-Water Equations

Continuous

$$\frac{\partial h}{\partial t} = -\nabla \bullet (h \underset{\sim}{V})$$

$$\frac{\partial}{\partial t}\underset{\sim}{V} = -\eta \underset{\sim}{k} \times \underset{\sim}{V} - \nabla[K + gh]$$

Discrete

$$\frac{\partial h_0}{\partial t} = -D_0[\bar{h}_c \underset{\sim}{V}_c]$$

$$\frac{\partial \underset{\sim}{V}_c}{\partial t} = -\bar{\eta}_c \underset{\sim}{k} \times \underset{\sim}{V}_c - \underset{\sim}{G}_c[K_0 + gh_0]$$

D_0 **= discrete divergence operator**

$\underset{\sim}{G}_c$ **= discrete gradient operator**

$\bar{h}_c$ **= averaging of mass to cell corners**

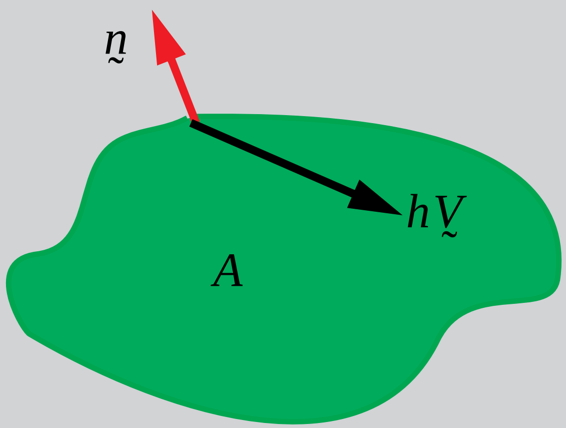
$\bar{\eta}_c$ **= averaging of vorticity to cell corners**

C_0 **= discrete curl operator**

The Discrete Divergence Operator

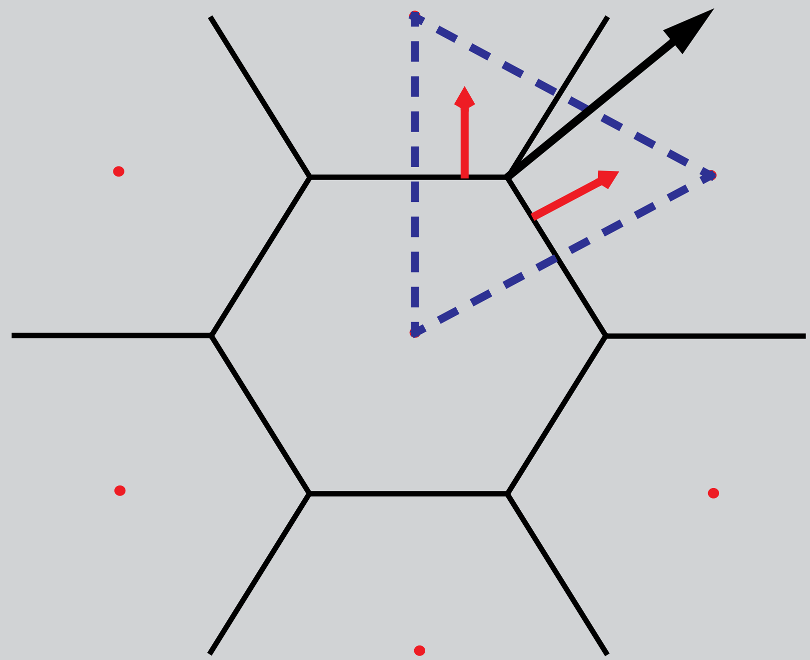
Definition

$$\nabla\bullet(h\underset{\sim}{V}) \equiv \lim_{A \to 0} \frac{1}{A}\int \underset{\sim}{n} \bullet h\underset{\sim}{V} \; dl$$



By Analogy

$$D_0(h\underset{\sim}{V}) = \frac{1}{A_0}\sum_{c=1}^{6}(\underset{\sim}{n}_c^{+} + \underset{\sim}{n}_c^{-}) \bullet \bar{h}_c \underset{\sim}{V}_c \left(\frac{\Delta t}{2}\right)$$



Conservation of Total Energy

Continuous Equations

$$K\left\{\frac{\partial h}{\partial t} = -\nabla\bullet(h\underset{\sim}{V})\right\}$$

$$gh\left\{\frac{\partial h}{\partial t} = -\nabla\bullet(h\underset{\sim}{V})\right\}$$

$$h\underset{\sim}{V}\bullet\left\{\frac{\partial}{\partial t}\underset{\sim}{V} = -\eta\underset{\sim}{k}\times\underset{\sim}{V} - \nabla[K+gh]\right\}$$

$$\int_A \frac{\partial}{\partial t}\left\{h\left[K+\frac{1}{2}gh\right]\right\}dA = 0$$

Discrete Equations

$$K_0\left\{\frac{\partial h_0}{\partial t} = -D_0[\bar{h}_c, \underset{\sim}{V}_c]\right\}$$

$$gh_0\left\{\frac{\partial h_0}{\partial t} = -D_0[\bar{h}_c, \underset{\sim}{V}_c]\right\}$$

$$h_0\sum_{c=1}^{6}\underset{\sim}{V}_c\bullet\left\{\frac{\partial \underset{\sim}{V}_c}{\partial t} = -\bar{\eta}_c\underset{\sim}{k}\times\underset{\sim}{V}_c - \underset{\sim}{G}_c[K_0+gh_0]\right\}$$

$$\cancel{\sum_{i=0}^{n}\frac{\partial}{\partial t}\left\{h_i\left[K_i+\frac{1}{2}gh_i\right]\right\}A_i = 0}$$

We use our degrees of freedom in G_c, K_0, and $\bar{h}_c$ to make this happen.

So what the are the forms of $\underset{\sim}{G}_c$, K_0, and $\bar{h}_c$?

The Gradient Operator

$$\underset{\sim}{G}_c(K) \bullet \underset{\sim}{e}_1 = \frac{\Delta n}{A_c}\left[\left(\frac{K_1 + K_2}{2}\right) - K_0\right]$$

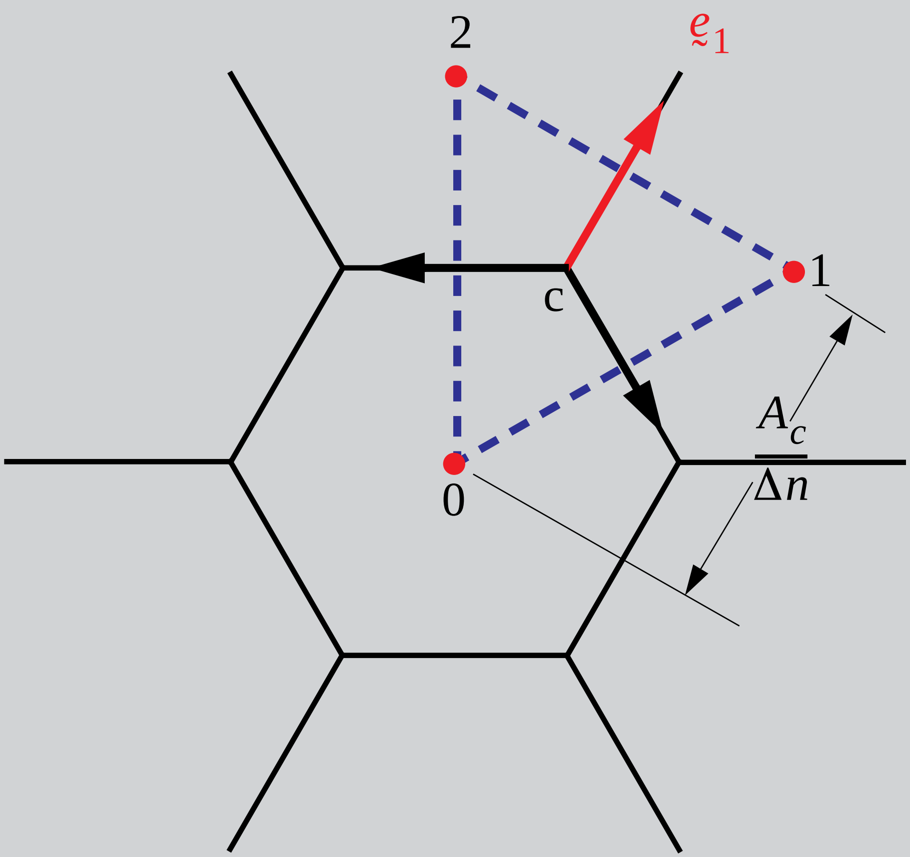
(other components look similar)

Kinetic Energy

$$K_0 = \frac{1}{6}\sum_{c=1}^{6}\frac{1}{2}(\underset{\sim}{V}_c \bullet \underset{\sim}{V}_c)$$

The Mass Averaging Operator

$$\bar{h}_c = \frac{1}{3}(h_0 + h_1 + h_2)$$



A 2-D turbulence simulation

Full shallow-water equations with a free surface

doubly-periodic f-plane, $f = 1.0 \times 10^{-4} s^{-1}$

Initial Conditions: $h = 400 \pm 50 m$, $\delta = \pm 5.0 \times 10^{-5} s^{-1}$, $\zeta = \pm 5.0 \times 10^{-5} s^{-1}$

Resolution: 128x128 grid, $\Delta n = 100 km$, $\lambda = 700 km$

Simulation 1

Purpose: address conservation

integration length 40 days
no dissipation

Simulation 2

Purpose: Energy/Enstrophy Cascade

integration length 1000 days
∇^6 **diffusion on velocity vector**

Simulation 1:

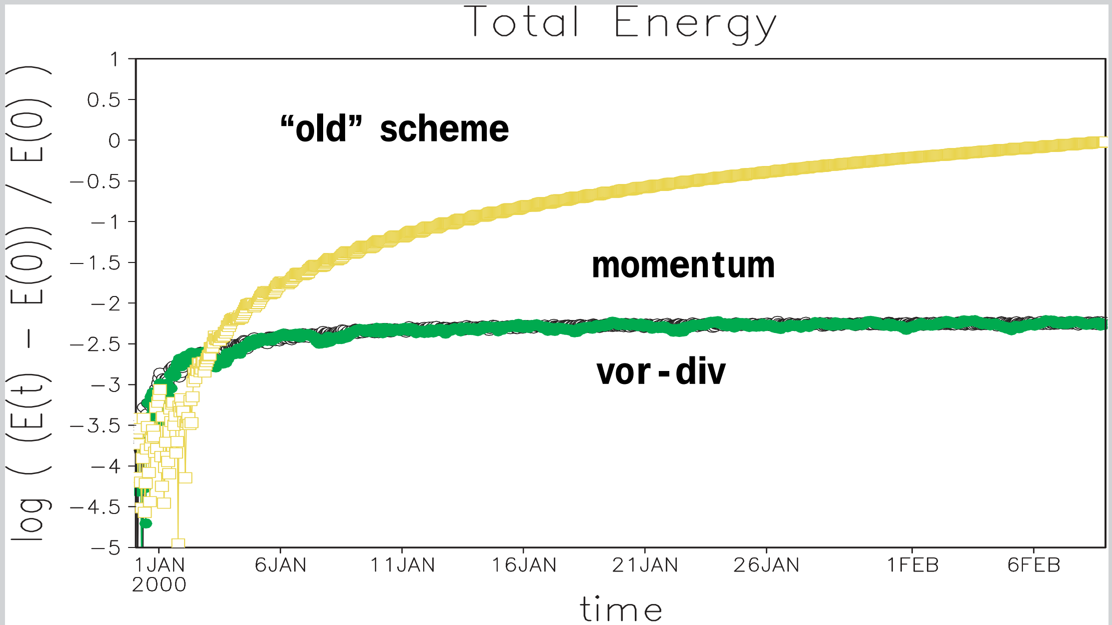
Comparison with “old” scheme

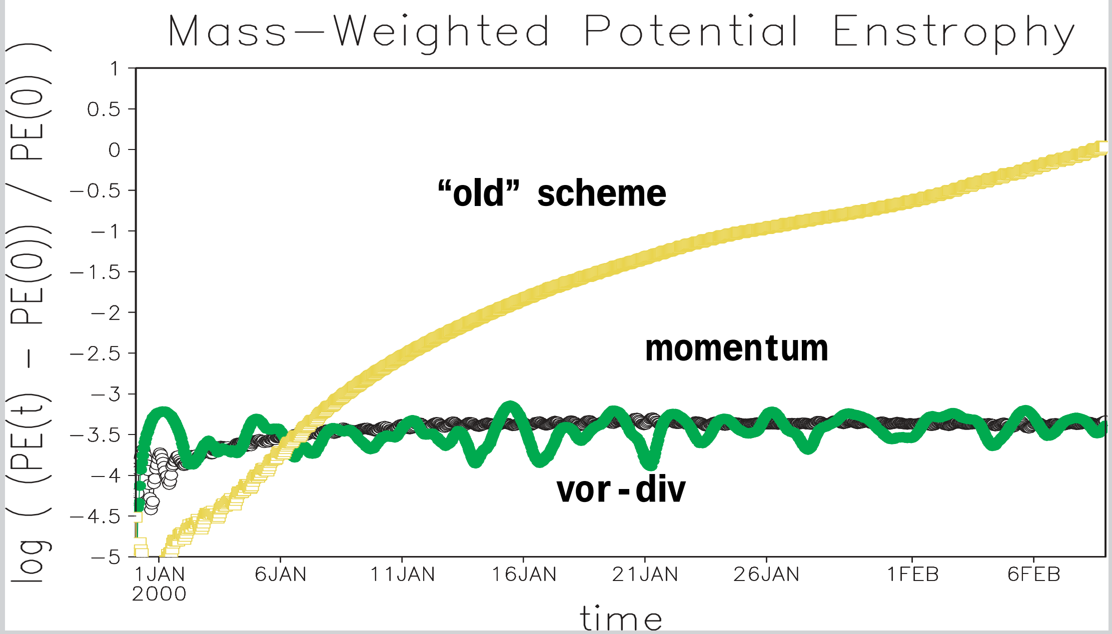
Top figure: the fractional change in total energy relative to the initial amount of energy.

Bottom figure: same as top, expect for potential enstrophy.

Over 40 days of integration, the “old” scheme shows an order one change in both total energy and potential enstrophy.

The new scheme shows a change of ~0,005% in total energy and ~0.0005% in potential enstrophy.



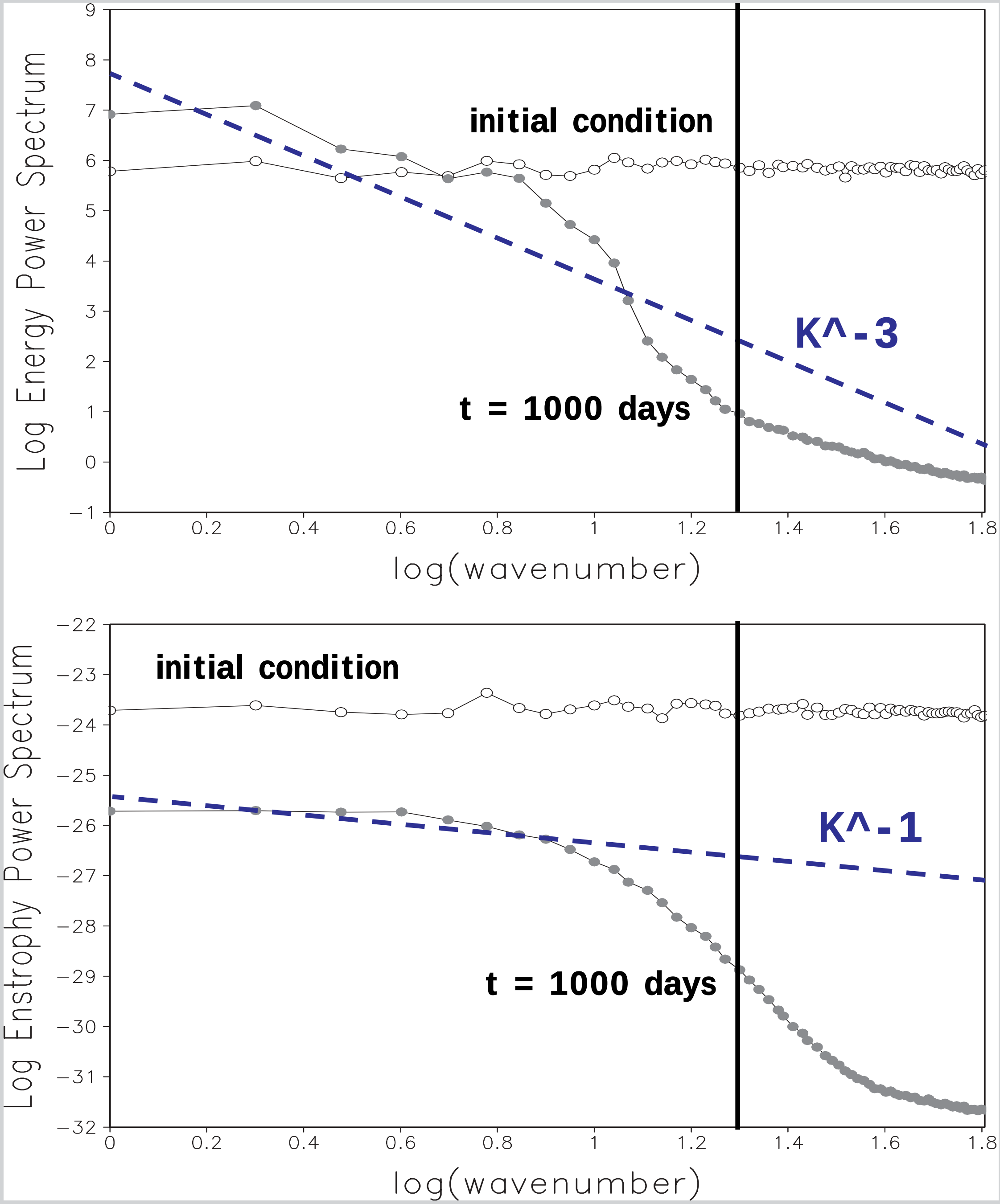


Simulation 2:
Cascade of KE and Enstrophy

KE spectrum (top)
Enstrophy spectrum (bottom)

Both figures show the spectrum at t=0 and t=1000 days.

Throughout most of the inertial range (scales larger than the Rossby radius), the correct cascade of kinetic energy and enstrophy is obtained.



Exchanging the Momentum Formulation for the Vorticity-Divergence Formulation

Continuous Equations

$$\nabla\bullet\left\{\frac{\partial}{\partial t}\underset{\sim}{V} = -\eta\underset{\sim}{k}\times\underset{\sim}{V} - \nabla[K+gh]\right\}$$

$$\nabla\times\left\{\frac{\partial}{\partial t}\underset{\sim}{V} = -\eta\underset{\sim}{k}\times\underset{\sim}{V} - \nabla[K+gh]\right\}$$

$$\frac{\partial\delta}{\partial t} = \underset{\sim}{k}\bullet\nabla\times(\eta\underset{\sim}{V}) - \nabla^2[K+gh]$$

$$\frac{\partial\eta}{\partial t} = -\nabla\bullet(\eta\underset{\sim}{V})$$

Discrete Equations

$$D_0\left\{\frac{\partial\underset{\sim}{V}_c}{\partial t} = -\bar{\eta}_c\underset{\sim}{k}\times\underset{\sim}{V}_c - \underset{\sim}{G}_c[K_0+gh_0]\right\}$$

$$C_0\left\{\frac{\partial\underset{\sim}{V}_c}{\partial t} = -\bar{\eta}_c\underset{\sim}{k}\times\underset{\sim}{V}_c - \underset{\sim}{G}_c[K_0+gh_0]\right\}$$

$$\frac{\partial\delta_0}{\partial t} = C_0(\eta\underset{\sim}{V}) - L_0[K+gh]$$

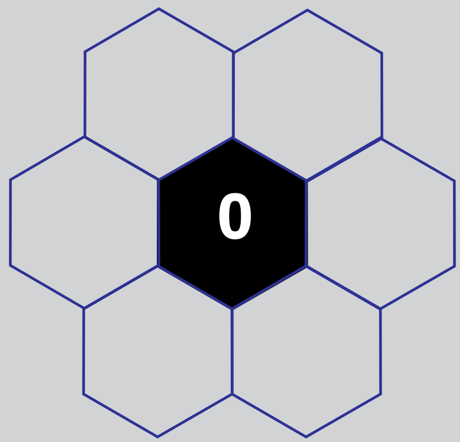
$$\frac{\partial\eta_0}{\partial t} = -D_0(\eta\underset{\sim}{V})$$

What does the discrete Laplacian look like?

The discrete Laplacian

The discrete forms of the divergence and gradient operators specify the form of the Laplacian operator.

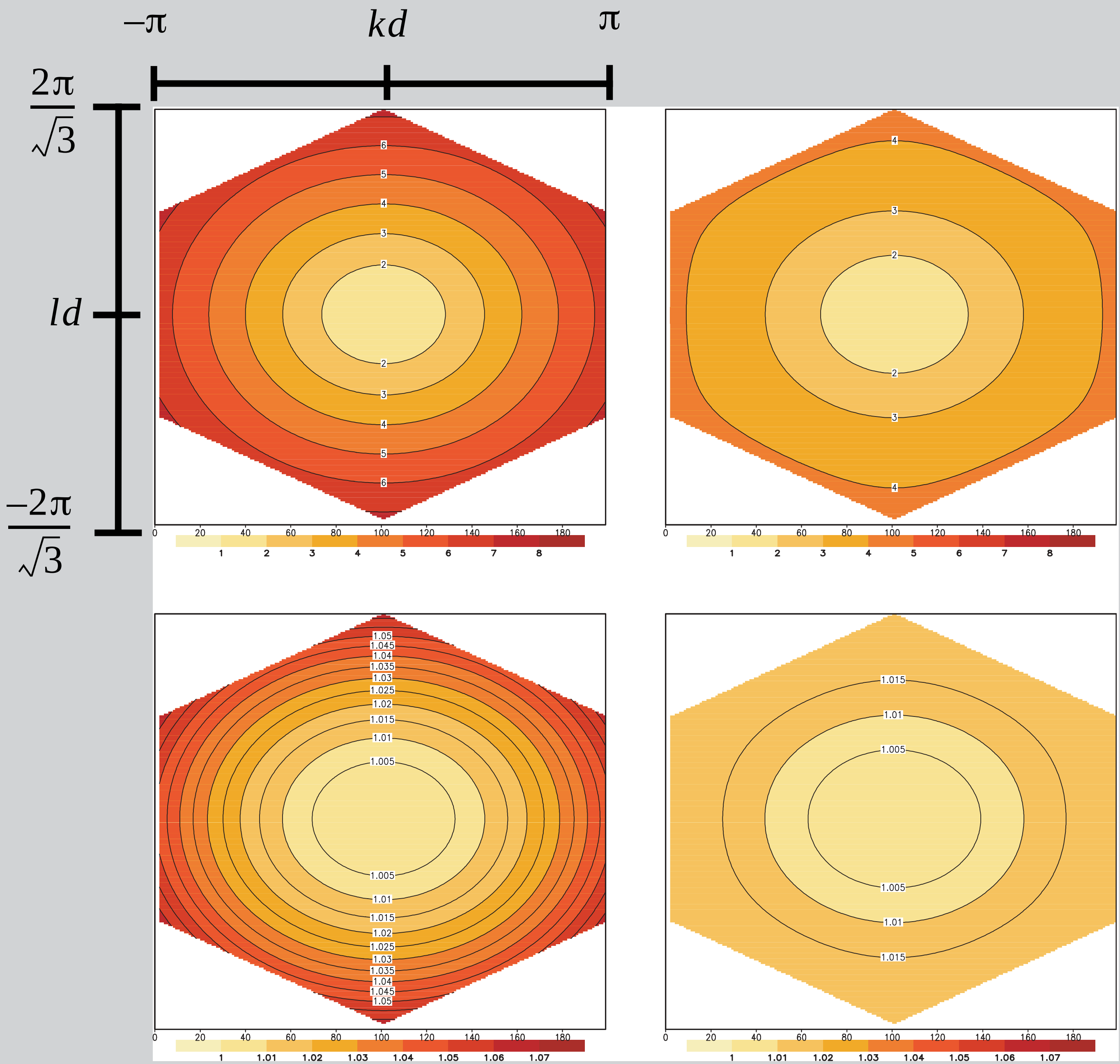
$$L(f_0) = \frac{f_1 + f_2 + f_3 + f_4 + f_5 + f_6 - 6f_0}{\sqrt{3}A_0}$$



This is the same form proposed by Masuda and Ohnishi (1986) and is consistent with the Z-grid (Randall, 1994)

Next, we will make use of the discrete Laplacian in two ways:

1) **Inertia-gravity wave dispersion relation**
2) **“Spectral” transforms directly from geodesic grids**



Rossby Radius resolved

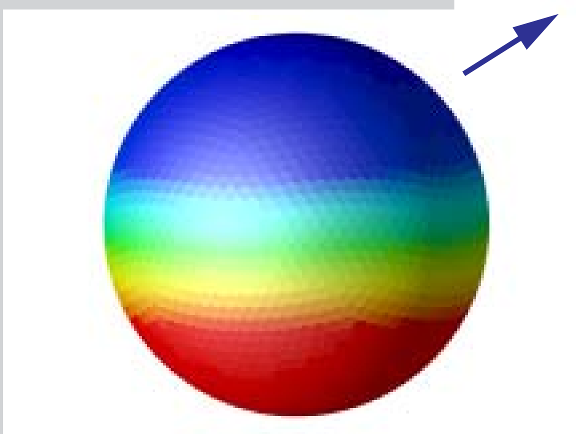
Rossby Radius not resolved

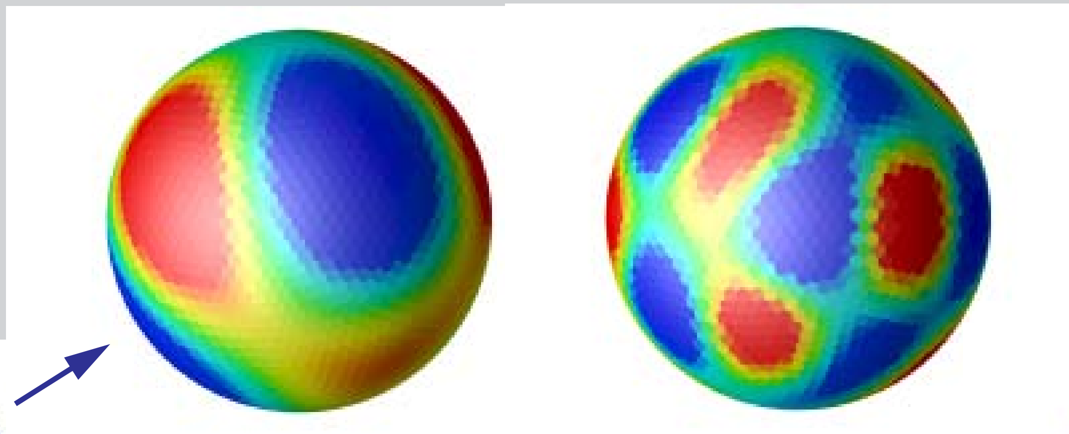
continuous

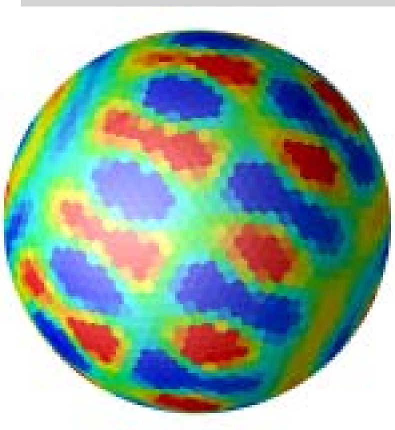
Z-grid

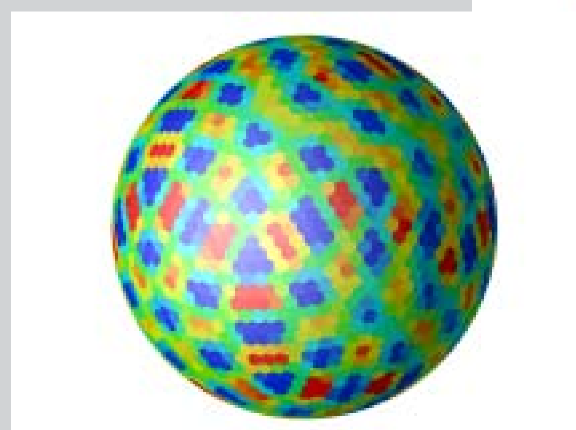
A “spectral” transform directly from the geodesic grid

We solve the eigenvalue problem $\nabla^2 f = \lambda f$.









The eigenvectors form a complete and orthogonal basis.

Moving to the primitive equations on the sphere

The Held-Suarez Test Case

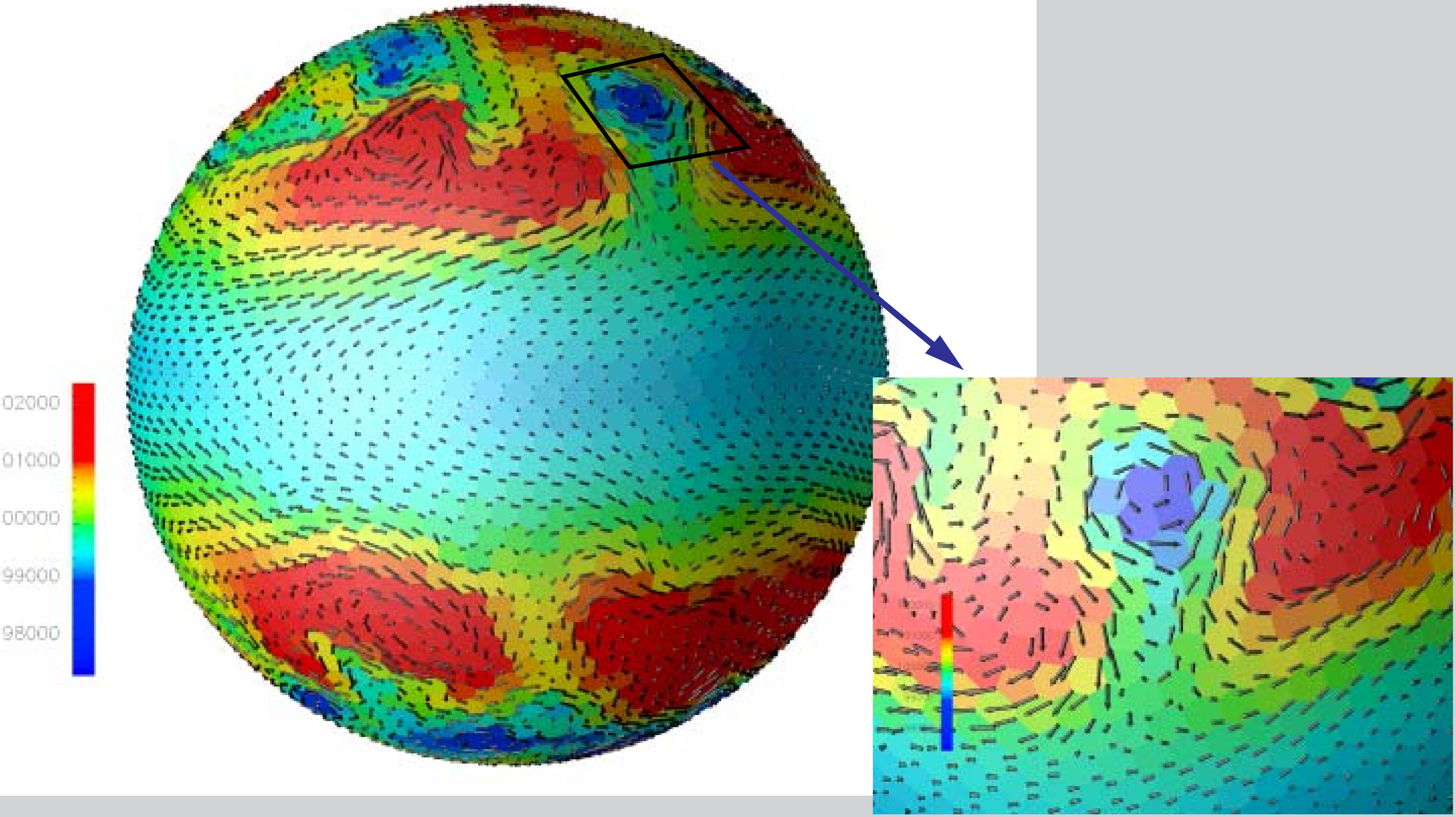
Zonally-symmetric thermal forcing “pushes” the atmosphere towards a baroclinicly-unstable state.

Linear drag on the velocity field mimics surface friction.

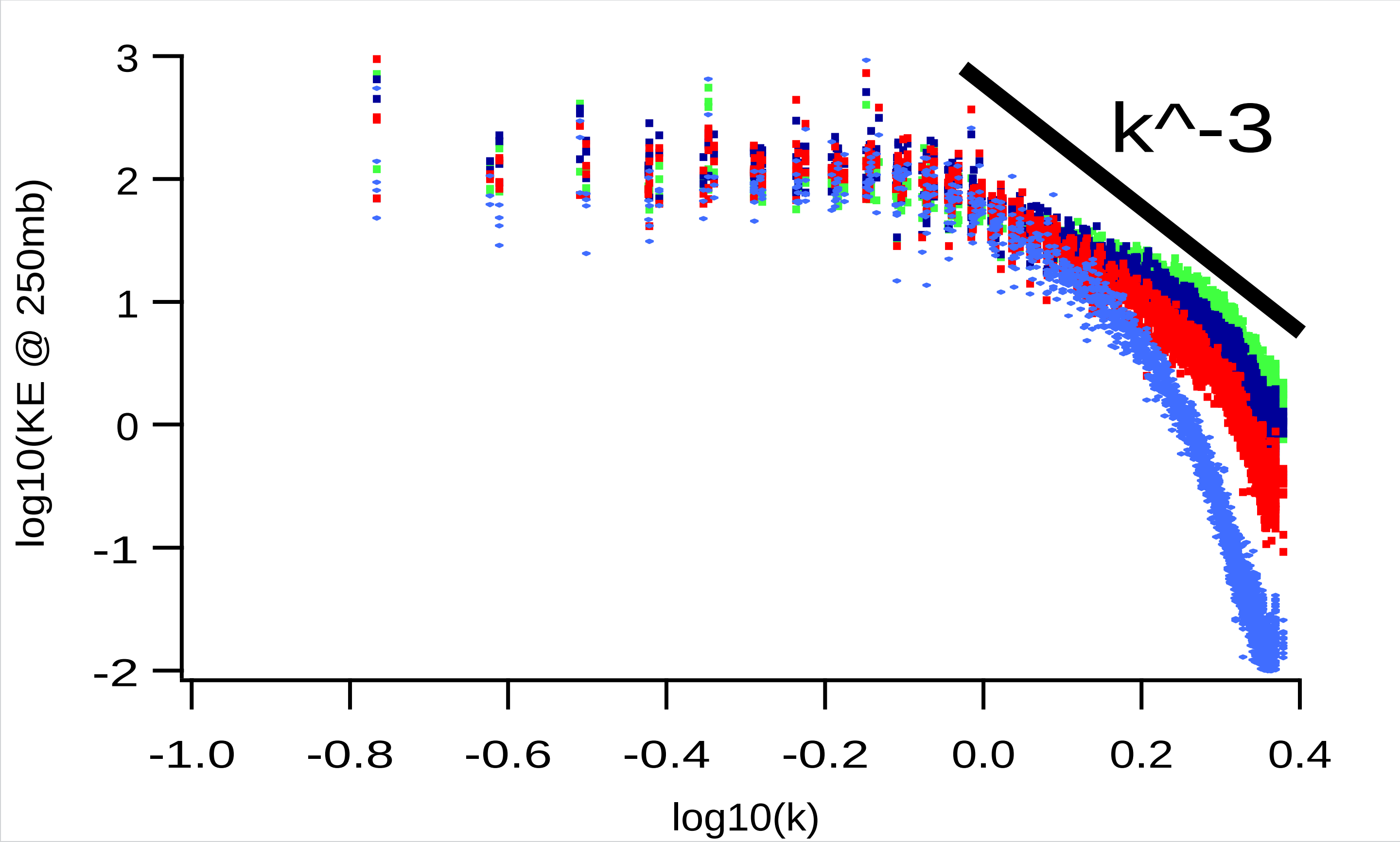
We will look at a snapshot in time, then at the spectrum of kinetic energy and enstrophy.

Held-Suarez Test Case/Momentum Formulation

a snapshot of surface pressure and velocity, μ = 1e14



Held-Suarez Test Case/Momentum Formulation
Spectra of Kinetic Energy per unit mass at 250mb
four values of μ: 1e12, 1e13, 1e14, 1e15 m4/s



Note! All of these results are using an extremely low resolution of 2562.

Held-Suarez Test Case/Momentum Formulation
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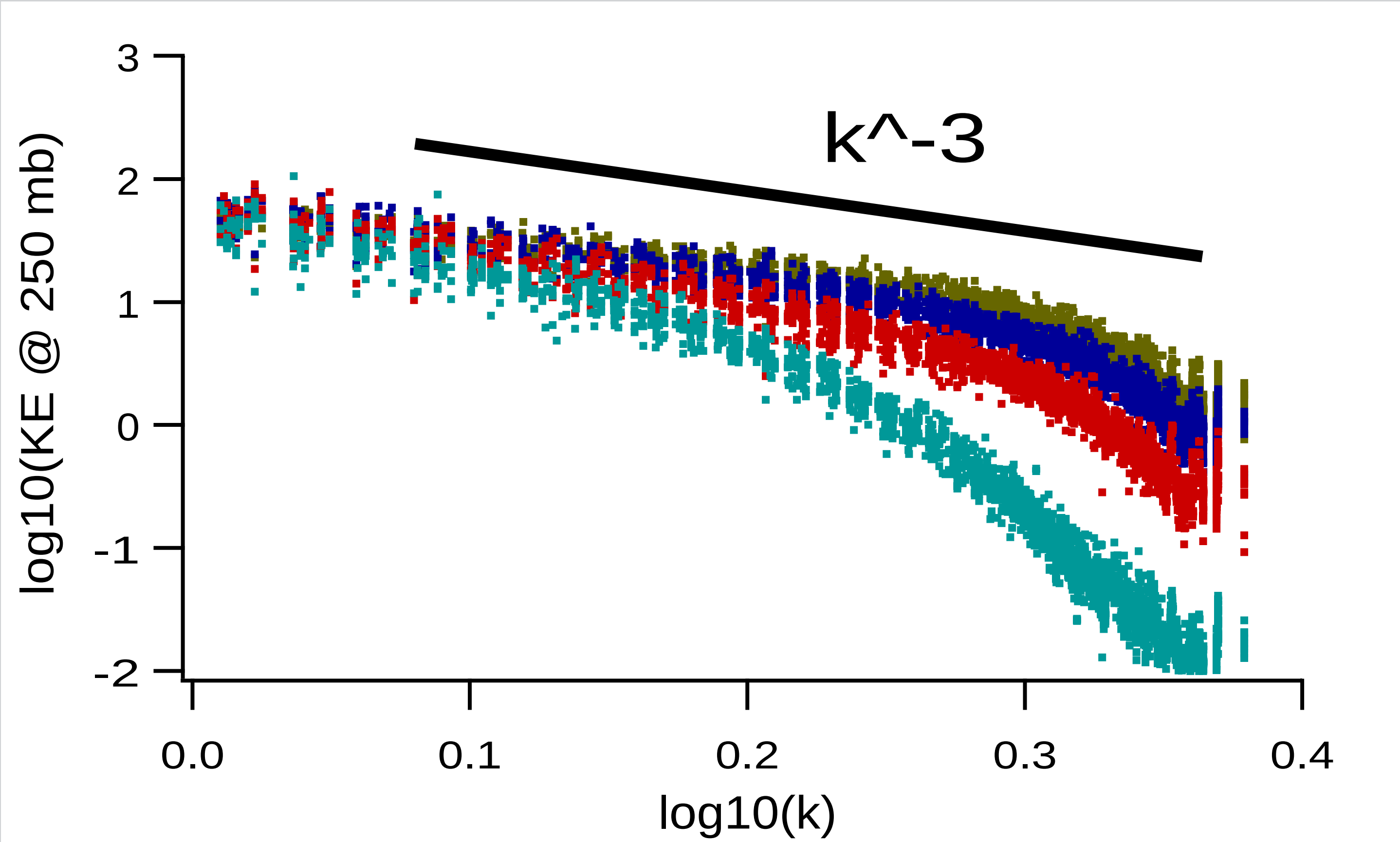
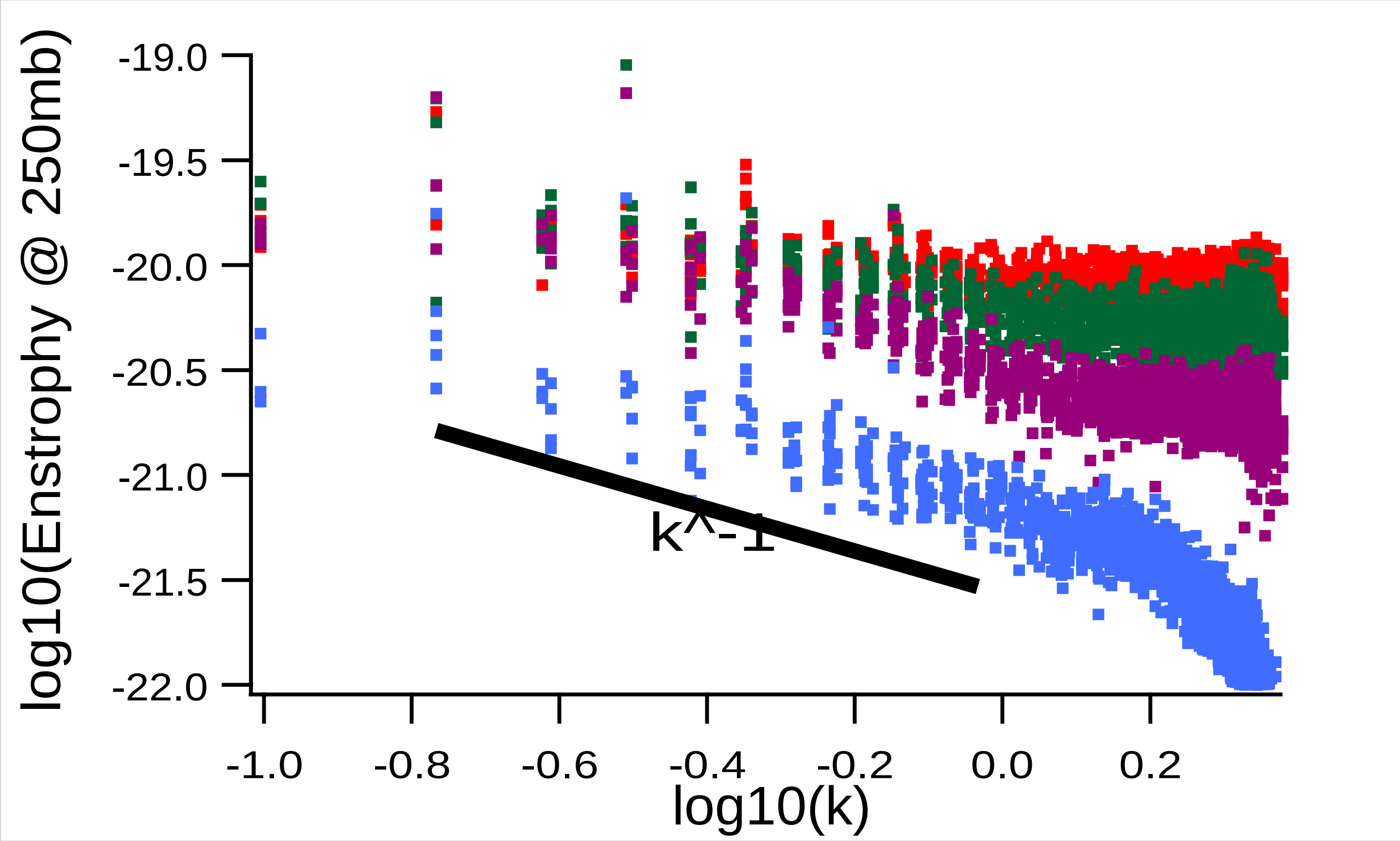


Held-Suarez Test Case/Momentum Formulation
Spectra of Enstrophy at 250mb

four values of μ: 1e12, 1e13, 1e14, 1e15 m4/s



Summary

We have derived a set of discrete operators (divergence, curl, and gradient) that mimic, in many important ways, their continuous counterparts.

As a result, the discrete system conserves total energy..... just like the continuous system.

Furthermore, the momentum formulation and the vorticity-divergence formulation are compatible..... just like the continuous system.

The Laplacian operator is specified by the forms of the divergence and gradient operators. This form leads to a superior simulation of the geostrophic adjustment process.

The numerical scheme produces the correct energy spectrum and enstrophy spectrum, even at low resolution.

We are testing out this numerical scheme in the CSU AGCM.

Manuscripts in press

Ringler and Randall, A potential enstrophy and energy conserving numerical scheme for solution of the shallow-water equations on a geodesic grid. MWR, in press.

Ringler and Randall, The ZM-grid: An alternative to the Z-grid. MWR, in press.

Information on the web

http://kiwi.atmos.colostate.edu/BUGS/projects/geodesic/